Math 564: Real analysis and measure theory
Lecture

Theorem. Finite Borel measures on Polish spaces are tight.

Proof (continued). For a Polish space X and a finite Borel measure μ on X, it is enough to show that for each $\varepsilon > 0$ there is a compact $K \subseteq X$ with $\mu(K) \approx_\varepsilon \mu(X)$. Fix $\varepsilon > 0$.

Let $\varepsilon_n := \frac{1}{n}$ and for each $n \in \mathbb{N}^+$, let $\{B_\ell^{\varepsilon_n}\}_{\ell \in \mathbb{N}}$ be a ctbl cover of X with closed balls of radius $\leq \varepsilon_n$ (such a cover exists by separability). Because $X = \bigcup\limits_{L \in \mathbb{N}}^{\uparrow} \left(\bigcup\limits_{\ell \leq L} B_\ell^{\varepsilon_n} \right)$, we have $\mu(X) \approx_{\varepsilon \cdot 2^{-(n+1)}} \mu\left(\bigcup\limits_{\ell \leq L_n} B_\ell^{\varepsilon_n} \right)$ for L_n large enough, by monotone convergence. Put $C_n := \bigcup\limits_{\ell \leq L_n} B_\ell^{\varepsilon_n}$, so C_n is closed and $K := \bigcap\limits_{n \in \mathbb{N}} C_n$ is still closed but also totally bounded by def, hence compact. Finally,

$$\mu(X \setminus K) \leq \mu\left(\bigcup_{n \in \mathbb{N}} X \setminus C_n \right) \leq \sum_{n \in \mathbb{N}} \varepsilon \cdot 2^{-(n+1)} = \varepsilon. \qquad \square$$

Cor (Strong regularity and tightness for locally fin. measures). Let X be a Polish space. Then every locally finite Borel measure on X is strongly regular and tight.

Proof. Polish spaces are 2^{nd} ctbl, so local finiteness is equiv. to σ-finiteness by open sets, i.e. $X = \bigcup\limits_{n \in \mathbb{N}} U_n$ with each U_n open and finite measure. Thus, μ is strongly regular by a previous result for metric spaces and we only need to show tightness. From DST (descriptive set theory), we know that open subsets of Polish spaces are Polish (with a different equivalent metric), so on each U_n, we know that μ is tight. We leave the rest of the proof as HW. $\square$

Measurable functions

Def. Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces. A function $f: X \to Y$ is said to be:

(a) $(\mathcal{I}, \mathcal{J})$-measurable if $f^{-1}(J) \in \mathcal{I}$ for every $J \in \mathcal{J}$.

(b) $\mathcal{I}$-measurable (or just measurable if $\mathcal{I}$ is clear from the context) if Y is a metric space and f is $(\mathcal{I}, \mathcal{B}(Y))$-measurable.

(c) Borel if X and Y are metric spaces and f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.

(d) μ-measurable if μ is a measure on $(X, \mathcal{I})$, Y is a metric space and f is Meas_μ-measurable, i.e. $f^{-1}(B)$ is μ-meas for each Borel $B \subseteq Y$.

Remark. For functions $f: \mathbb{R} \to \mathbb{R}$ we will view the left $\mathbb{R}$ as the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and the right $\mathbb{R}$ as a metric space, so the definition of λ-measurable (or Lebesgue measurable) is asymmetric: $f^{-1}(\text{Borel})$ is λ-measurable. This done because to get more functions be called measurable since the theory works for them.

Prop. Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces and $f: X \to Y$. If for some $\mathcal{J}_0 \subseteq \mathcal{J}$ which generates $\mathcal{J}$ as a σ-algebra, we have $f^{-1}(J_0) \in \mathcal{I}$ for all $J_0 \in \mathcal{J}_0$, then f is $(\mathcal{I}, \mathcal{J})$-measurable.

Proof. Let $\mathcal{S} := \{ J \in \mathcal{J} : f^{-1}(J) \in \mathcal{I} \}$ and observe that $\mathcal{S} \supseteq \mathcal{J}_0$ and $\mathcal{S}$ is a σ-algebra since preimages respect unions and complements. Hence $\mathcal{S} = \mathcal{J}$. $\square$

Cor. Let $(X, \mathcal{I})$ be a measurable space, Y be a metric space, and let $f: X \to Y$. If $f^{-1}(V) \in \mathcal{I}$ for each open $V \subseteq Y$, then f is $\mathcal{I}$-measurable. In particular, continuous functions are Borel because $f^{-1}(\text{open})$ is open.

The following is one of the reasons for building measure theory.

Theorem. Pointwise limits of measurable functions are measurable. More precisely, if $(X, \mathcal{I})$ is a measurable space and Y is a separable metric space (e.g. $\mathbb{R}$) then $\lim_{n \to \infty} f_n$ is $\mathcal{I}$-measurable for any sequence of $\mathcal{I}$-measurable functions $f_n : X \to Y$ for which $\lim_{n \to \infty} f_n(x)$ exists for each $x \in X$.

Proof. By the last corollary, it is enough to show that $f^{-1}(U) \in \mathcal{I}$ for each open $U \subseteq Y$. Note that the openness of U gives the following: for $x \in X$,

$$f(x) \in U \implies \forall^\infty_n \, f_n(x) \in U \qquad (\forall^\infty_n := \exists m \, \forall n \geq m)$$

If the converse were also true, we would be done because then

$$f^{-1}(U) = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} f_n^{-1}(U)$$

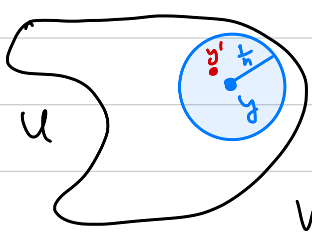
so $f^{-1}(U) \in \mathcal{I}$.

But the converse isn't true, for example, let $U = (0,1) \subseteq \mathbb{R}$ and $f_n(x) := \frac{1}{n}$, so $f_n(x) \in U$ for all n, but the limit is $0 \notin U$. The converse holds for closed sets but U is open. However, using separability, we can find a presentation of U, which behaves as both open and closed:

Claim. $\bigcup_{k \in \mathbb{N}} V_k = U = \bigcup_{k \in \mathbb{N}} \overline{V_k}$ for some open $V_k \subseteq Y$.

Proof. Let $D \subseteq Y$ be a ctbl dense set and let $\mathcal{V} := \{B_{1/n}(y) : y \in D, n \in \mathbb{N}^+, \overline{B_{1/n}(y)} \subseteq U\}$, so $\mathcal{V}$ is ctbl. Note that if $V \in \mathcal{V}$, then $\overline{V} \subseteq U$ by definition, so it is enough to show that $U = \bigcup_{V \in \mathcal{V}} V$. Fix $y \in U$, hence $\overline{B_{1/n}(y)} \subseteq U$ for large enough $n \in \mathbb{N}$. Let $y' \in D$ such that $y' \in B_{1/2n}(y)$, equiv. $y \in B_{1/2n}(y')$.

But $\overline{B_{1/2n}(y')} \subseteq \overline{B_{1/n}(y)} \subseteq U$, so $B_{1/2n}(y') \in \mathcal{V}$. Hence $y \in \bigcup_{V \in \mathcal{V}} V$.



□ (Claim)

We now finally have: $\forall x \in X$, $f(x) \in U \iff \exists k \, \forall^\infty_n \, f_n(x) \in \overline{V_k}$.

Proof of $\Rightarrow$: $f(x) \in U = \bigcup_{k \in \mathbb{N}} V_k \implies \exists k \; f(x) \in V_k \implies \exists k \; \forall^\infty_n \, f_n(x) \in V_k$.

Proof of $\Leftarrow$: $\exists k \; \forall^\infty_n \, f_n(x) \in \overline{V_k} \implies \exists k \; f(x) \in \overline{V_k}$ (by closedness of $\overline{V_k}$) $\implies f(x) \in U$.

Thus, $f^{-1}(U) = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} f_n^{-1}(\overline{V_k}) \in \mathcal{I}$. □

Prop. Let X, Y be metric spaces, where Y is 2nd ctbl (e.g. Polish). Let f be a strong-

ly regular Borel measure on X (e.g. a finite measure or σ-finite by open sets).
Let $f: X \to Y$ be a μ-measurable function. Then

(a) f is Borel on a conull Borel set, i.e. $f|_{X'} : X' \to Y$ is a Borel function for some conull Borel X'. (Note: $\mathcal{B}(X') = \{B \in \mathcal{B}(X) : B \subseteq X'\}$.)

(b) Luzin's theorem. $\forall \varepsilon > 0$, f is continuous on a closed set C with $\mu(X \setminus C) < \varepsilon$, i.e. $f|_C : C \to Y$ is continuous.

Proof. Let $\{V_n\}_{n \in \mathbb{N}}$ be ctbl basis for Y, so it generates $\mathcal{B}(Y)$ as a σ-algebra.

(a) $f^{-1}(V_n)$ is μ-measurable, hence $f^{-1}(V_n) =_\mu B_n$ for some Borel $B_n \subseteq X$.
Let $Z := \bigcup_{n \in \mathbb{N}} (f^{-1}(V_n) \Delta B_n)$, so Z is null, hence $Z \subseteq \tilde{Z}$ where $\tilde{Z}$ is Borel and still null. Put $X' := X \setminus \tilde{Z}$, so X' is Borel and conull. Then $(f|_{X'})^{-1}(V_n) = f^{-1}(V_n) \cap X' = B_n \cap X'$, which is Borel.
So $f|_{X'}$ is Borel. $\boxed{a}$

(b) $f^{-1}(V_n)$ is μ-measurable, hence by strong outer regularity, $\exists\, U_n \subseteq X$ open such that $d_\mu(U_n, f^{-1}(V_n)) := \mu(U_n \Delta f^{-1}(V_n)) \leq \varepsilon \cdot 2^{-(n+2)}$ (in fact, $U_n \supseteq f^{-1}(V_n)$, but we don't need this). Let $Z := \bigcup_{n \in \mathbb{N}} U_n \Delta f^{-1}(V_n)$ so $\mu(Z) \leq \sum_{n \in \mathbb{N}} \varepsilon \cdot 2^{-(n+2)} = \frac{\varepsilon}{2}$. Again by outer regularity, there is an open set $\tilde{Z} \supseteq Z$ with $\mu(\tilde{Z} \setminus Z) \leq \varepsilon/2$, so $\mu(\tilde{Z}) \leq \varepsilon$.
Take $C := X \setminus \tilde{Z}$, so it's closed and $\mu(X \setminus C) \leq \varepsilon$. Furthermore:

$$(f|_C)^{-1}(V_n) = f^{-1}(V_n) \cap C = U_n \cap C,$$

so $(f|_C)^{-1}(V_n)$ is open relative to C and hence $f|_C$ is continuous. $\boxed{b}$